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On Certain Mean Values of Polynomials on the Unit Interval

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For any continuous function $f\colon [-1, 1] \mapsto \mathbb{C}$ and any $p \in (0, \infty)$, let $\|f\|_p := (2^{-1} \int_{-1}^{1} |f(x)|^p \, dx)^{1/p}$; in addition, let $\|f\|_\infty := \max_{-1 \leqslant x \leqslant 1} |f(x)|$. It is known that if f is a polynomial of degree n, then for all $p > 0$,

$$\|f\|_\infty \leqslant C_p \, n^{2/p} \, \|f\|_p,$$

where C_p is a constant depending on p but not on n. In this result of Nikolskiĭ (1951), which was independently obtained by Szegö and Zygmund (1954), the order of magnitude of the bound is the best possible. We obtain a sharp version of this inequality for polynomials not vanishing in the open unit disk. As an application we prove the following result. If f is a real polynomial of degree n such that $f(-1) = f(1) = 0$ and $f(z) \neq 0$ in the open unit disk, then for $p > 0$ the quantity $\|f'\|_\infty / \|f\|_p$ is maximized by polynomials of the form $c(1+x)^{n-1}(1-x)$, $c(1+x)(1-x)^{n-1}$, where $c \in \mathbb{R} \backslash \{0\}$. This extends an inequality of Erdős (1940).

1. INTRODUCTION AND STATEMENT OF RESULTS

For any continuous function $f\colon [-1, 1] \mapsto \mathbb{C}$ and any $p \in (0, \infty)$ let

$$\|f\|_p := \left(\frac{1}{2} \int_{-1}^{1} |f(x)|^p \, dx \right)^{1/p};$$

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in addition, let

$$\|f\|_\infty := \max_{-1\leqslant x\leqslant 1} |f(x)|.$$

It is known (see [7, Sect. 6.8]) that $\|f\|_p$ tends to the limit

$$\exp\left(\tfrac{1}{2}\int_{-1}^{1} \log |f(x)|\, dx\right)$$

as $p\to 0$. This is exactly the value given to the functional $\|f\|_p$ when $p=0$.

It was proved by Erdős and Grünwald [5, Theorem III] that if f is a polynomial having only real zeros and $-1, 1$ as consecutive zeros, then $\|f\|_1 \leqslant (2/3)\,\|f\|_\infty$. Considering the polynomial $1-x^2$ we see that the inequality is sharp. Mentioning $(1-x^2)^n$ as an example they remarked [5, p. 358] that the same ratio may assume values less than any preassigned number howsoever small. We may still ask for the *precise* lower bound for $\|f\|_1/\|f\|_\infty$ if the degree of f does not exceed a fixed integer n. It turns out that this ratio is minimized by polynomials of the form $c(1+x)(1-x)^{n-1}$ and $c(1+x)^{n-1}(1-x)$, where $c\neq 0$. In fact, we shall consider the ratio $\|f\|_p/\|f\|_\infty$ for an arbitrary $p\geqslant 0$.

Let $\mathscr{F}_n$ be the class of all polynomials of degree at most n. We say that $f\in\mathscr{P}_n$ if

(i) $f\in\mathscr{F}_n$;

(ii) $f(z)\neq 0$ for $|z|<1$;

(iii) $f(x)>0$ for $-1<x<1$.

Given $\mu\in\{0, ..., [n/2]\}$, the set of all polynomials in $\mathscr{P}_n$ which have zeros of multiplicity at least μ at -1 and 1 will be denoted by $\mathscr{P}_{n,\mu}$. Note that $\mathscr{P}_{n,0}$ is the same as $\mathscr{P}_n$.

For $n\in\mathbb{N}$, $\mu\in\{0, ..., [n/2]\}$ and $p\in[0,\infty)$, let

$$\mathfrak{M}_{n,\mu,p} := \inf\{\|f\|_p : f\in\mathscr{P}_{n,\mu},\ \|f\|_\infty = 1\}. \tag{1}$$

Furthermore for $k\in\{0, ..., n\}$, let

$$q_{n,k}(x) := (1+x)^k\,(1-x)^{n-k}, \qquad q_{n,k,*}(x) := \frac{n^n q_{n,k}(x)}{2^n k^k (n-k)^{n-k}}. \tag{2}$$

Note that $\|q_{n,k,*}\|_\infty = 1$.

We prove

THEOREM 1. *Let f be a polynomial of degree at most n with real coefficients and having no zeros in the open unit disk. Suppose, in addition, that*

f has zeros of multiplicity at least μ at -1 and 1, where $0 \leqslant \mu \leqslant [n/2]$. If f is not a constant multiple of $q_{n,\mu}$ or of $q_{n,n-\mu}$, then

$$\|f\|_p > \|q_{n,\mu,*}\|_p \, \|f\|_\infty \qquad (0 \leqslant p < \infty).$$

The analogue of the inequality of Nikolskiĭ, and Szegö and Zygmund, for polynomials not vanishing in $|z| < 1$, is contained in the following simple consequence of Theorem 1.

Corollary 1. *Let f be a polynomial of degree at most n having no zeros in the open unit disk but whose coefficients may be nonreal. Suppose, in addition, that $f(z) := (1-z^2)^\mu \, g(z)$, where $0 \leqslant \mu \leqslant [n/2]$ and g is a polynomial of degree at most $n - 2\mu$. Then for $0 \leqslant p < \infty$, we have*

$$\|f\|_\infty \leqslant \frac{\|f\|_p}{\|q_{n,\mu,*}\|_p},$$

where equality holds only for constant multiples of $q_{n,\mu,*}$ and $q_{n,n-\mu,*}$.

Inequality (4) can also be written as

$$\|f\|_\infty \leqslant \begin{cases} \dfrac{\mu^\mu (n-\mu)^{n-\mu}}{n^n} \left(\dfrac{\Gamma(pn+2)}{\Gamma(\mu p+1)\, \Gamma((n-\mu)\, p+1)} \right)^{1/p} \|f\|_p, & 0 < p < \infty \\ \dfrac{\mu^\mu (n-\mu)^{n-\mu}}{n^n} \, e^n \, \|f\|_p, & p = 0, \end{cases}$$

where μ is as in Corollary 1.

Here is another consequence of Theorem 1.

Corollary 2. *Let f be a real polynomial of degree at most n, such that $f(-1) = f(1) = 0$ and $f(z) \neq 0$ for $|z| < 1$. If f is not a constant multiple of $q_{n,1}$ or of $q_{n,n-1}$, then*

$$\|f'\|_\infty < \frac{\|q'_{n,1}\|_\infty}{\|q_{n,1}\|_p} \, \|f\|_p \qquad (0 \leqslant p < \infty).$$

This corollary is an extension of a result of Erdős [5, p. 310].

2. PROOF OF THEOREM 1

For the proof of Theorem 1, we shall assume that $f(x) > 0$ for $-1 < x < 1$ and $\|f\|_\infty = 1$. We shall show that for each $\mu \in \{0, ..., [n/2]\}$ and $0 \leqslant p < \infty$, the infimum $\mathfrak{M}_{n,\mu,p}$ defined in (1) is attained only when f is

$q_{n,\mu,*}$ or $q_{n,n-\mu,*}$. The proof of Theorem 1 is rather long, and so we shall present it as a sequence of lemmas and connecting paragraphs.

2.1. *Preparatory Lemmas*

LEMMA 1. *Given n, μ, and p as above, there exists a polynomial F belonging to $\mathscr{P}_{n,\mu}$ with $\|F\|_\infty = 1$ such that $\|F\|_p = \mathfrak{M}_{n,\mu,p}$.*

Proof. If $f(z) := \sum_{v=0}^{n} a_v z^v$ belongs to $\mathscr{P}_{n,\mu}$ and $\|f\|_\infty = 1$, then

$$|a_v| \leqslant \binom{n}{v} \qquad \text{for} \quad 0 \leqslant v \leqslant n.$$

Indeed, $f(z)$ can be expressed as $a_0 \prod_{v=1}^{n} (1 - \zeta_v z)$, where $|\zeta_v| \leqslant 1$ for $1 \leqslant v \leqslant n$ and so

$$|a_v| \leqslant a_0 \binom{n}{v} = f(0) \binom{n}{v} \leqslant \binom{n}{v}.$$

Note in addition that

$$1 = \max_{-1 \leqslant x \leqslant 1} f(x) \leqslant a_0 \sum_{v=0}^{n} \binom{n}{v} = 2^n a_0;$$

i.e.,

$$a_0 \geqslant 2^{-n}.$$

For each positive integer m there exists a polynomial

$$h_m(z) := \sum_{v=0}^{n} a_{v,m} z^v$$

belonging to $\mathscr{P}_{n,\mu}$ with $\|h_m\|_\infty = 1$ such that

$$\|h_m\|_p < \mathfrak{M}_{n,\mu,p} + m^{-1}.$$

Since $|a_{v,m}| \leqslant \binom{n}{v}$ for all $m \in \mathbb{N}$ and $0 \leqslant v \leqslant n$, we can use a standard argument to select a subsequence $\{h_{m_1}, ..., h_{m_k}, ...\}$ of $\{h_m\}$ converging uniformly on any compact subset of $\mathbb{C}$ to a polynomial F in $\mathscr{F}_n$. Since $h_m(0) \geqslant 2^{-n}$ for each m we note that F *cannot be identically zero*. Hence, by a well-known theorem of Hurwitz [1, p. 176], F cannot have any zeros in $|z| < 1$ although it must have zeros of multiplicity at least μ at -1 and 1.

Hence, the limiting polynomial F belongs to $\mathscr{P}_{n,\mu}$. As regards the subsequence $h_{m_1}, ..., h_{m_k}, ...,$ we could have assumed (by choosing a further subsequence if necessary) that if ξ_k is the point of $[-1, 1]$ where h_{m_k} takes the value 1, then $\xi_1, ..., \xi_k, ...$ tends to a point ξ^*. Using the mean value theorem and a well-known inequality of A. Markov, according to which $\|h'\|_\infty \leqslant n^2 \|h\|_\infty$ for every polynomial h of degree at most n, we conclude that $F(\xi^*) = 1$. Thus, $\|F\|_\infty$ is equal to 1 since, obviously, it cannot be larger than 1.

LEMMA 2. *If $F \in \mathscr{P}_{n,\mu}$ and $\|F\|_p = \mathfrak{M}_{n,\mu,p}$, then the zeros of F must be all real.*

Proof. Let us suppose that

$$F(z) := G(z)(z - a - ib)(z - a + ib),$$

where $a, b \in \mathbb{R}$, $b \neq 0$, $a^2 + b^2 \geqslant 1$. Let ξ be a point in $[-1, 1]$ where F assumes the value 1 and consider the polynomial

$$\begin{aligned} F(\varepsilon; z) &:= F(z) - \varepsilon G(z)(z - \xi)^2 \\ &= G(z)\{(1-\varepsilon)\, z^2 - 2(a - \varepsilon\xi)\, z + a^2 + b^2 - \varepsilon\xi^2\}. \end{aligned}$$

For small positive ε the zeros of the quadratic $(1-\varepsilon)\, z^2 - 2(a - \varepsilon\xi)\, z + a^2 + b^2 - \varepsilon\xi^2$ are complex and the product of their moduli is $(a^2 + b^2 - \varepsilon\xi^2)/(1-\varepsilon)$, which is greater than or equal to 1. For such values of ε, the polynomial $F(\varepsilon; \cdot)$ belongs to $\mathscr{P}_{n,\mu}$ and $\|F(\varepsilon; \cdot)\|_\infty = F(\varepsilon; \xi) = 1$. However, $\|F(\varepsilon; \cdot)\|_p < \|F\|_p$, which is a contradiction.

Remark 1. In Lemma 2 we have shown that the polynomial F cannot have non-real zeros. So, while looking for a polynomial in $\mathscr{P}_{n,\mu}$ for which $\mathfrak{M}_{n,\mu,p}$ is attained, we only need to examine those whose zeros are all real.

We shall say that $f \in \wp_{n,\mu}$ if

- $f \in \mathscr{P}_{n,\mu}$;
- the zeros of f are all real;
- $\|f\|_\infty = 1$.

According to Lemma 2,

$$\mathfrak{M}_{n,\mu,p} = \inf\{\|f\|_p : f \in \wp_{n,\mu}\}. \tag{6}$$

It is a simple consequence of Rolle's theorem that a polynomial with only real zeros has only one critical point between two consecutive zeros. So, each polynomial $f \in \wp_{n,\mu}$ attains the value 1 at exactly one point in $[-1, 1]$, which we shall *always* denote by ξ.

LEMMA 3. *Let $f \in \wp_{n,\mu}$. If ξ belongs to $[-1, 1]$ and*

$$f(\xi) = \max_{-1 \leqslant x \leqslant 1} f(x),$$

then $|\xi| \leqslant 1 - 2\mu/n$.

Proof. There is nothing to prove when $\mu = 0$; so, let $\mu \geqslant 1$. Due to obvious symmetry, it is enough to prove that $\xi \notin (1 - 2\mu/n, 1)$. Clearly, $f'(\xi)$ must be zero. If $f(x) := c(x-1)^{\mu} \prod_{\nu=1}^{n-\mu} (x - x_{\nu})$, then $f'(\xi)$ can vanish only if

$$A(\xi) := \sum_{\nu=1}^{n-\mu} \frac{1}{\xi - x_{\nu}} - \frac{\mu}{1-\xi}$$

does. But $1/(\xi - x_{\nu}) \leqslant 1/(1+\xi)$ for $1 \leqslant \nu \leqslant n - \mu$. Hence

$$A(\xi) \leqslant \frac{n-\mu}{1+\xi} - \frac{\mu}{1-\xi} = \frac{n - 2\mu - n\xi}{1-\xi^2} < 0 \qquad \text{if} \quad \xi \in \left(1 - \frac{2\mu}{n}, 1\right).$$

LEMMA 4. *Let $F \in \wp_{n,\mu}$ and $\|F\|_p = \mathfrak{M}_{n,\mu,p}$. Then*

$$F(x) := c(1-x)^j (1+x)^k (1+\alpha x) \qquad (c > 0,\ j + k = n - 1,\ -1 \leqslant \alpha \leqslant 1).$$

In addition, $j \geqslant \mu$ or $j \geqslant \mu - 1$ according to whether $\alpha \in (-1, 1]$ or $\alpha = -1$ and $k \geqslant \mu$ or $k \geqslant \mu - 1$ according to whether $\alpha \in [-1, 1)$ or $\alpha = 1$.

Proof. Let ξ be the point of $[-1, 1]$ where F attains the value 1. First we observe that F cannot have zeros in $(-\infty, -1)$ and $(1, \infty)$ at the same time. Suppose it does. Let λ_1 be the smallest zero of F and λ_m the largest. It is easily seen that for all small $\varepsilon > 0$ the polynomial

$$F_{\varepsilon,1}(x) := F(x) + \varepsilon \frac{F(x)}{(x - \lambda_1)(x - \lambda_m)} (x - \xi)^2$$

belongs to $\wp_{n,\mu}$ and $F_{\varepsilon,1}(x) \leqslant F(x)$ for all $x \in [-1, 1]$, the inequality being strict in $(-1, 1) \backslash \{\xi\}$. So F may have zeros in $(-\infty, -1)$ or in $(1, \infty)$ but not in both.

Assume that F has no zeros in $(-\infty, -1)$. We claim that F cannot have two or more distinct zeros in $(1, \infty)$. Suppose it does. Let λ_m be the largest zero and λ_l the largest but one. It is geometrically evident that for all small $\varepsilon > 0$, the polynomial

$$F_{\varepsilon, 2}(x) := F(x) - \varepsilon \frac{F(x)}{(x-\lambda_l)(x-\lambda_m)} (x-\xi)^2$$

belongs to $\wp_{n,\mu}$ and $F_{\varepsilon,2}(x) \leqslant F(x)$ for all $x \in [-1, 1]$, the inequality being strict in $(-1, 1)\backslash\{\xi\}$. So F can have at most one distinct zero in $(-\infty, -1) \cup (1, \infty)$.

Suppose that F has a zero λ_m in $(1, \infty)$. We claim that λ_m cannot be a multiple zero. Suppose it is. Then for all small $\varepsilon > 0$, the polynomial

$$\begin{aligned} F_{\varepsilon, 3}(x) &:= F(x) - \varepsilon \frac{F(x)}{(x-\lambda_m)^2} (x-\xi)^2 \\ &= \frac{F(x)}{(x-\lambda_m)^2} \{(1-\varepsilon)\, x^2 - 2(\lambda_m - \varepsilon\xi)\, x + \lambda_m^2 - \varepsilon\xi^2\} \end{aligned}$$

belongs to $\wp_{n,\mu}$. Indeed, $F_{\varepsilon,3}(\xi) = F(\xi) = 1$ and there exists $\varepsilon^* > 0$ such that for all $\varepsilon \in (0, \varepsilon^*)$ the quadratic $(1-\varepsilon)\, x^2 - 2(\lambda_m - \varepsilon\xi)\, x + \lambda_m^2 - \varepsilon\xi^2$ has two different real zeros, both lying in $(1, \infty)$. In addition, $F_{\varepsilon,3}(x) < F(x)$ for all $x \in (-1, 1)\backslash\{\xi\}$. So, if F has a zero in $(-\infty, -1) \cup (1, \infty)$, it should be simple.

We have proved that F must be of the form

$$F(x) := c(1-x)^j\, (1+x)^k\, (1+\alpha x)$$

with $c > 0$, $j + k \leqslant n-1$ and $-1 \leqslant \alpha \leqslant 1$. In addition, $j \geqslant \mu$ or $j \geqslant \mu - 1$ according to whether $\alpha \in (-1, 1]$ or $\alpha = -1$ and $k \geqslant \mu$ or $k \geqslant \mu - 1$ according as $\alpha \in [-1, 1)$ or $\alpha = 1$. We claim that the sum of the multiplicities of the zeros of F at -1 and 1 cannot be less than $n-1$. Suppose it is. First let $\alpha \in (-1, 0) \cup (0, 1)$. The polynomial

$$F_{\varepsilon, 4}(x) := F(x) - \varepsilon \frac{F(x)}{(1+\alpha x)} (x-\xi)^2$$

belongs to $\wp_{n,\mu}$ for all small $\varepsilon > 0$. Furthermore, $F_{\varepsilon,4}(x) < F(x)$ for all $x \in (-1, 1)\backslash\{\xi\}$. If $\alpha \in \{-1, 0, 1\}$ then we have to prove that $F(x)$ cannot be of the form $c(1-x)^j\, (1+x)^k$ with $j + k \leqslant n-2$. For this we consider the polynomial

$$F_{\varepsilon, 5}(x) := F(x) - \varepsilon F(x)(x-\xi)^2,$$

which is of degree at most n, and obtain a contradiction.

We say that a polynomial f belongs to $\pi_{n,\mu}$ if

- it is of the form

$$f(x) := c(1+x)^k (1-x)^{n-k-1} (1+\alpha x),$$

where $0 \leqslant k \leqslant n-1$, $-1 \leqslant \alpha \leqslant 1$, $c > 0$;

- it has zeros of *multiplicity at least* μ at -1 and $+1$;
- $\|f\|_\infty = 1$.

Lemma 4 in conjunction with Lemma 2 says that while looking for a polynomial in $\mathscr{P}_{n,\mu}$ for which $\mathfrak{M}_{n,\mu,p}$ is attained, we may restrict our search to those which belong to $\pi_{n,\mu}$. In other words,

$$\mathfrak{M}_{n,\mu,p} = \inf\{\|f\|_p : f \in \pi_{n,\mu}\}. \tag{7}$$

Given $n \in \mathbb{N}$, $\mu \in \{0, ..., [n/2]\}$ and $\xi \in [-1+2\mu/n, 1-2\mu/n]$, we say that $f \in \pi_{n,\mu,\xi}$ if $f \in \pi_{n,\mu}$ and $f(\xi) = 1$. Let

$$\mathfrak{M}_{n,\mu,p,\xi} := \inf\{\|f\|_p : f \in \pi_{n,\mu,\xi}\}, \qquad -1+\frac{2\mu}{n} \leqslant \xi \leqslant 1-\frac{2\mu}{n}. \tag{8}$$

Then, clearly

$$\mathfrak{M}_{n,\mu,p} = \inf_{|\xi| \leqslant 1-2\mu/n} \mathfrak{M}_{n,\mu,p,\xi} = \inf_{0 \leqslant \xi \leqslant 1-2\mu/n} \mathfrak{M}_{n,\mu,p,\xi}. \tag{9}$$

For $0 \leqslant k \leqslant n-1$, let

$$\xi_{1,n,k} := -1+\frac{2k}{n}, \qquad \xi_{2,n,k} := -1+\frac{2k+2}{n} \tag{10}$$

and $I_{n,k} := [\xi_{1,n,k}, \xi_{2,n,k}]$. The following lemma helps us to identify the elements of $\pi_{n,\mu,\xi}$.

Lemma 5. *Let $n \geqslant 3$ and $1 \leqslant k \leqslant n-2$. For each ξ in $I_{n,k}$ there exists one and only one $\alpha = \alpha(\xi)$ in $[-1, 1]$ such that the derivative of*

$$P_{n,k}(\alpha; x) := (1+x)^k (1-x)^{n-1-k} (1+\alpha x)$$

vanishes at ξ. Moreover, $\alpha(\xi)$ increases strictly from -1 to 1 as ξ increases from one end of the interval $[\xi_{1,n,k}, \xi_{2,n,k}]$ to the other.

Proof. The derivative of $P_{n,k}(\alpha; \cdot)$ with respect to x vanishes at ξ if and only if

$$\alpha = \alpha(\xi) := \frac{(n-1)\xi + (n-2k-1)}{1-(n-2k-1)\xi - n\xi^2}.$$

We show that $|\alpha(\xi)| \leqslant 1$ if $\xi \in I_{n,k}$. Setting

$$g_n(\xi) := n\xi^2 + (n-2k-1)\,\xi - 1$$

we see that

$$g_n(-1) = 2k > 0, \qquad g_n\left(-1+\frac{2k}{n}\right) = -\frac{2k}{n} < 0,$$

$$g_n\left(-1+\frac{2k+2}{n}\right) = -\frac{2}{n}(n-k-1) < 0, \qquad g_n(1) = 2(n-k-1) > 0.$$

Hence g_n has a zero in $(-1, -1+2k/n)$ and also in $(-1+(2k+2)/n, 1)$. Consequently, it cannot have any zero in $I_{n,k}$. This implies that $\alpha(\xi)$ is a well-defined real number for all ξ in $I_{n,k}$. Elementary calculations show that $\alpha(\xi) \leqslant 1$ for $\xi \in I_{n,k}$ if and only if $(1+\xi)(\xi+1-(2k+2)/n) \leqslant 0$, which is certainly true for all ξ in $I_{n,k}$. In addition, $-1 \leqslant \alpha(\xi)$ for $\xi \in I_{n,k}$ if and only if $(1-\xi)(\xi+1-2k/n) \geqslant 0$ and so for all $\xi \in I_{n,k}$. Thus, we have proved that $-1 \leqslant \alpha(\xi) \leqslant 1$ for all $\xi \in I_{n,k}$.

As can be easily verified, $\alpha(\xi_{1,n,k}) = -1$ and $\alpha(\xi_{2,n,k}) = 1$. We have to show that that $\alpha(\xi)$ increases strictly from -1 to $+1$ as ξ increases from one end of the interval $I_{n,k}$ to the other. For all $\xi \in I_{n,k}$ we have

$$\alpha'(\xi) = \frac{(n\xi + n - 2k - 1)^2 + n - 1 - n\xi^2}{\{1-(n-2k-1)\,\xi - n\xi^2\}^2}. \tag{11}$$

Hence $\alpha'(\xi) > 0$ if $n-1-n\xi^2 > 0$, which certainly holds if $|\xi| \leqslant 1-1/n$. Since $I_{n,k} \subset [-1+1/n, 1-1/n]$ it follows that $\alpha'(\xi) > 0$ for all $\xi \in I_{n,k}$.

Remark 2. In Lemma 5, we have proved that for each ξ in $I_{n,k}$, $1 \leqslant k \leqslant n-2$ there exists one and only one $\alpha \in [-1, 1]$ such that

$$\frac{\partial}{\partial x} P_{n,k}(\alpha; x)\Big|_{x=\xi} = 0,$$

which is a necessary condition for the maximum of $cP_{n,k}(\alpha;\cdot)$ to be attained at ξ. It follows that for any given ξ in $I_{n,k}$, $1\leqslant k\leqslant n-2$ the set $\pi_{n,k,\xi}$ contains just one element, namely the polynomial

$$\begin{aligned} P_{n,k,\xi}(x) &:= \frac{1}{P_{n,k}(\alpha;\xi)}\, P_{n,k}(\alpha;x), \\ \alpha(\xi) &:= \frac{(n-1)\xi+(n-2k-1)}{1-(n-2k-1)\xi-n\xi^2}. \end{aligned} \tag{12}$$

As k varies from μ to $n-\mu-1$ the intervals $I_{n,k}$ cover the interval $[-1+2\mu/n, 1-2\mu/n]$. Using the obvious symmetry we conclude that *for each* ξ *in* $[-1+2\mu/n, 1-2\mu/n]$, $1\leqslant\mu\leqslant[n/2]$ *the set* $\pi_{n,\mu,\xi}$ *has one and only one element. The same can be said for* ξ *in* $(-1, -1+2/n)\cup(1-2/n, 1)$ *when* $\mu=0$. In fact, simple calculations show that for any ξ in $(-1, -1+2/n)$ the set $\pi_{n,0,\xi}$ contains the polynomial

$$\begin{aligned} P_{n,0,\xi}(x) &:= \left(\frac{1-x}{1-\xi}\right)^{n-1} \frac{n\xi-1-(n-1)x}{\xi-1}, \\ &\qquad -1<\xi<-1+\frac{2}{n} \end{aligned} \tag{13}$$

and no other; for ξ in $(1-2/n, 1)$ the only element of $\pi_{n,0,\xi}$ is the polynomial

$$\begin{aligned} P_{n,n-1,\xi}(x) &:= \left(\frac{1+x}{1+\xi}\right)^{n-1} \frac{n\xi+1-(n-1)x}{\xi+1}, \\ &\qquad 1-\frac{2}{n}<\xi<1. \end{aligned} \tag{14}$$

It may be added that for $-1<\xi<-1+2/n$ we have

$$P_{n,0,\xi}(x) = \frac{(1-x)^{n-1}\,(1+\alpha(\xi)x)}{(1-\xi)^{n-1}\,(1+\alpha(\xi)\xi)},$$

where $\alpha(\xi):=-(n-1)/(n\xi-1)$ increases from $(n-1)/(n+1)$ to 1 as ξ increases from -1 to $-1+2/n$. For $1-2/n<\xi<1$ we have

$$P_{n,n-1,\xi}(x) = \frac{(1+x)^{n-1}\,(1+\alpha(\xi)x)}{(1+\xi)^{n-1}\,(1+\alpha(\xi)\xi)},$$

where $\alpha(\xi):=-(n-1)/(n\xi+1)$ increases from -1 to $-(n-1)/(n+1)$ as ξ increases from $1-2/n$ to 1.

Remark 3. For each $\xi \in [-1, 1]$ there is only one $k \in \{0, ..., n-1\}$ such that $\xi \in I_{n,k}$ except when ξ is of the form $-1 + 2k/n$. In the latter case ξ belongs to $I_{n,k}$ for two consecutive values of k; however, there is no ambiguity in the definition of $P_{n,k,\xi}$ because $P_{n,k,\xi}$ for $\xi = \xi_{2,n,k}$ and $P_{n,k+1,\xi}$ for $\xi = \xi_{1,n,k+1}$ are the same.

DEFINITION. Given $n \in \mathbb{N}$, $\mu \in \{0, ..., [n/2]\}$, $p \in [0, \infty)$ and ξ in $[-1 + 2\mu/n, 1 - 2\mu/n]$ let us denote by $\mathscr{E}_{n,\mu,\xi}$ the set of all polynomials f in $\pi_{n,\mu,\xi}$ such that $\|f\|_p = \mathfrak{M}_{n,\mu,p,\xi}$.

Remark 4. It follows from above that for $1 \leqslant \mu \leqslant [n/2]$ and any ξ in $[-1 + 2\mu/n, 1 - 2\mu/n]$ the set $\mathscr{E}_{n,\mu,\xi}$ consists of only one element, namely $P_{n,k,\xi}$ with $k \in \{\mu, ..., n-\mu-1\}$ such that $\xi \in I_{n,k}$. The same is true of $\mathscr{E}_{n,0,\xi}$, except possibly for $\xi = \pm 1$.

What can we say about $\pi_{n,0,1}$ and $\pi_{n,0,-1}$? For this we note that a polynomial of the form

$$f(x) := c(1+x)^k (1-x)^{n-1-k} (1+\alpha x), \qquad -1 \leqslant \alpha \leqslant 1, \quad f(1) = 1,$$

assumes its maximum on $[-1, 1]$ at 1 if and only if

$$f(x) = f_\alpha(x) := \left(\frac{1+x}{2}\right)^{n-1} \frac{1+\alpha x}{1+\alpha}, \qquad -\frac{n-1}{n+1} \leqslant \alpha \leqslant 1.$$

It is easily checked that if $\alpha < \alpha'$ than $0 < f_{\alpha'}(x) < f_\alpha(x)$ for all $x \in (-1, 1)$. Hence $\|f_\alpha\|_p$ is a strictly decreasing function of α in $[-(n-1)/(n+1), 1]$. This implies that $\mathscr{E}_{n,0,1}$ consists of just one polynomial, namely

$$P^*_{n,n-1,1}(x) := \left(\frac{1+x}{2}\right)^n. \tag{15}$$

Similarly, $\mathscr{E}_{n,0,-1}$ has only one element, namely the polynomial

$$P^*_{n,0,-1}(x) := \left(\frac{1-x}{2}\right)^n. \tag{16}$$

Remark 5. We conclude that for all $p \in [0, \infty)$ the value of $\mathfrak{M}_{n,\mu,p,\xi}$ is determined as follows.

(i) First let $\xi \in [-1 + 2/n, 1 - 2/n]$. Then, as is easily seen, f can belong to $\pi_{n,\mu}$ only if $\mu \in \{1, ..., [n/2]\}$. Furthermore, $\xi \in I_{n,k} \subset [-1 + 2\mu/n, 1 - 2\mu/n]$, for some $k \in \{1, ..., n-2\}$ and

$$\mathfrak{M}_{n,\mu,p,\xi} = \|P_{n,k,\xi}\|_p, \tag{17}$$

where $P_{n,k,\xi}$ is as in (12);

(ii) if $\xi \in (-1, -1+2/n) \cup (1-2/n, 1)$, then

$$\mathfrak{M}_{n,0,p,\xi} = \|P_{n,0,\xi}\|_p \qquad \text{or} \quad \mathfrak{M}_{n,0,p,\xi} = \|P_{n,n-1,\xi}\|_p, \tag{18}$$

according to whether ξ lies in $(-1, -1+2/n)$ or in $(1-2/n, 1)$, respectively;

(iii) finally for $\xi = \pm 1$ we have

$$\mathfrak{M}_{n,0,p,1} = \|P^*_{n,n-1,1}\|_p, \qquad \mathfrak{M}_{n,0,p,-1} = \|P^*_{n,0,-1}\|_p, \tag{19}$$

where $P^*_{n,n-1,1}$ and $P^*_{n,0,-1}$ are as in (15) and (16), respectively.

2.2. *The Case $p>0$ and $\mu \geqslant 1$ of Theorem* 1

First we will find $\mathfrak{M}_{n,\mu,p}$ for $p>0$ and $\mu \geqslant 1$. Let us set

$$\Phi_p(\xi) := \int_{-1}^{1} \left| \frac{(1-x)^j (1+x)^k (1+\alpha(\xi)x)}{(1-\xi)^j (1+\xi)^k (1+\alpha(\xi)\xi)} \right|^p dx,$$

where $k \in \{\mu, ..., n-1-\mu\}$, $j = n-1-k$, $p>0$. Then from statement (i) of Remark 5 we have

$$\mathfrak{M}_{n,\mu,\xi,p} = \left(\tfrac{1}{2}\Phi_p(\xi)\right)^{1/p} \qquad (\xi \in I_{n,k} \subset [-1+2\mu/n, 1-2\mu/n]).$$

In order to determine

$$\mathfrak{M}_{n,\mu,p}, \qquad \mu \geqslant 1$$

we shall study, in view of (9), the behaviour of $\Phi_p(\xi)$ over the subintervals $I_{n,k} = [\xi_{1,n,k}, \xi_{2,n,k}]$ $(k = \mu, ..., n-\mu-1)$ of $[-1+2\mu/n, 1-2\mu/n]$. Because of obvious symmetry we may assume $k \geqslant j$ $(=n-1-k)$. We remind the reader that $\alpha(\xi_{1,n,k}) = -1$, $\alpha(\xi_{2,n,k}) = 1$ and that there is one and only one point

$$\xi^* = \xi^*_{n,k} := \frac{k-j}{j+k} = \frac{2k-(n-1)}{n-1} \tag{20}$$

in $I_{n,k}$ such that $\alpha(\xi^*) = 0$.

We shall end up with the conclusion

$$\min_{\xi_{1,n,k} \leqslant \xi \leqslant \xi_{2,n,k}} \Phi_p(\xi) = \min\{\Phi_p(\xi_{1,n,k}), \Phi_p(\xi_{2,n,k})\}. \tag{21}$$

The function Φ_p, whose definition depends on n as well as on k is differentiable at each interior point of $I_{n,k}$. At $\xi_{1,n,k}$ the right-hand derivative exists and at $\xi_{2,n,k}$ the left-hand derivative exists. So, $\Phi'_p(\xi_1)$ is to be understood as $\Phi'_p(\xi_1+)$ and $\Phi'_p(\xi_2)$ as $\Phi'_p(\xi_2-)$. As we shall see, Φ_p has at most two critical points in $I_{n,k} = [\xi_{1,n,k}, \xi_{2,n,k}]$ but only one point of local extremum. It lies in $(\xi_{1,n,k}, \xi_{2,n,k})$ and is a point of local maximum. A straightforward calculation gives

$$\frac{\Phi_p'(\xi)}{\Phi_p(\xi)} = p\alpha'(\xi) \left\{ \frac{\int_{-1}^{1} (1-x)^{jp} (1+x)^{kp} (1+\alpha(\xi)x)^{p-1} x \, dx}{\int_{-1}^{1} (1-x)^{jp} (1+x)^{kp} (1+\alpha(\xi)x)^{p} \, dx} - \frac{\xi}{1+\alpha(\xi)\,\xi} \right\}. \tag{22}$$

It is important to know the sign of $\Phi_p'(\xi)$ at the points

$$\xi_1 = \xi_{1,n,k} = \frac{k-j-1}{j+k+1}, \qquad \xi^* = \frac{k-j}{j+k}, \qquad \xi_2 = \xi_{2,n,k} = \frac{k-j+1}{j+k+1}.$$

For this we need the following well-known formula.

Lemma 6 [4, pp. 212–214]. *If* $\Re(a) > 0$ *and* $\Re(b) > 0$, *then*

$$\int_{-1}^{1} (1-t)^{a-1} (1+t)^{b-1} \, dt = 2^{a+b-1} \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}. \tag{23}$$

The quantity $\Phi_p'(\xi)/\Phi_p(\xi)$ can be explicitly calculated at the points ξ_1, ξ^*, ξ_2 since $\alpha(\xi_1) = -1$, $\alpha(\xi^*) = 0$, $\alpha(\xi_2) = 1$. Writing x in the form $-(1-x)+1$ we obtain

$$\begin{aligned}
\frac{\Phi_p'(\xi_1+)}{\Phi_p(\xi_1)} &= p\alpha'(\xi_1+) \left\{ \frac{\int_{-1}^{1} (1-x)^{jp+p-1} (1+x)^{kp} \, x \, dx}{\int_{-1}^{1} (1-x)^{jp+p} (1+x)^{kp} \, dx} - \frac{\xi_1}{1-\xi_1} \right\} \\
&= p\alpha'(\xi_1+) \left\{ -1 + \frac{\int_{-1}^{1} (1-x)^{jp+p-1} (1+x)^{kp} \, dx}{\int_{-1}^{1} (1-x)^{jp+p} (1+x)^{kp} \, dx} - \frac{\xi_1}{1-\xi_1} \right\} \\
&= p\alpha'(\xi_1+) \left\{ -1 + \frac{1}{2} \frac{(j+k+1)\,p+1}{(j+1)\,p} - \frac{\xi_1}{1-\xi_1} \right\} \qquad \text{by Lemma 6} \\
&= \frac{\alpha'(\xi_1+)}{2(j+1)},
\end{aligned}$$

where we have used the fact that $\xi_1 = (k-j-1)/(j+k+1)$. As noted in the proof of Lemma 5, $\alpha'(\xi) > 0$ for all ξ in $[-1+2/n, 1-2/n]$; hence $\Phi_p'(\xi_1+) > 0$. Obviously then there exists $\delta_1 > 0$ such that

$$\Phi_p'(\xi) > 0 \qquad \text{for} \quad \xi_1 \leqslant \xi < \xi_1 + \delta_1. \tag{24}$$

Since $\alpha(\xi^*)=0$ we get

$$\begin{aligned}\frac{\Phi_p'(\xi^*)}{\Phi_p(\xi^*)}&=p\alpha'(\xi^*)\left\{\frac{\int_{-1}^{1}(1-x)^{jp}(1+x)^{kp}\,x\,dx}{\int_{-1}^{1}(1-x)^{jp}(1+x)^{kp}\,dx}-\xi^*\right\}\\&=p\alpha'(\xi^*)\left\{-1+\frac{\int_{-1}^{1}(1-x)^{jp}(1+x)^{kp+1}\,dx}{\int_{-1}^{1}(1-x)^{jp}(1+x)^{kp}\,dx}-\xi^*\right\}\\&=p\alpha'(\xi^*)\left\{\frac{(k-j)\,p}{jp+kp+2}-\frac{k-j}{j+k}\right\}\\&=-p\alpha'(\xi^*)\,\frac{2(k-j)}{(jp+kp+2)(j+k)},\end{aligned}$$

wherein we have used (23) and the fact that $\xi^*=(k-j)/(j+k)$. Hence,

$$\Phi_p'(\xi^*)<0 \quad \text{if } j<k, \qquad \Phi_p'(\xi^*)=0 \quad \text{if } j=k. \tag{25}$$

Similarly, using the fact that $\alpha(\xi_2)=1$, we obtain

$$\frac{\Phi_p'(\xi_2-)}{\Phi_p(\xi_2)}=-\frac{\alpha'(\xi_2-)}{2(k+1)}.$$

There exists therefore a positive number δ_2 such that

$$\Phi_p'(\xi)<0 \qquad \text{for} \quad \xi_2-\delta_2<\xi\leqslant\xi_2. \tag{26}$$

Since Φ_p is an increasing function of ξ in $[\xi_1, \xi_1+\delta_1)$ and a decreasing function of ξ in $(\xi_2-\delta_2, \xi_2]$, it must have at least one critical point in (ξ_1, ξ_2). Let $\{\mathfrak{c}\}_{n,k}$ be the set of all its critical points in (ξ_1, ξ_2). Our argument will show that $\{\mathfrak{c}\}_{n,k}$ contains at most two points and that only one of them is a point of local extremum. The point of local extremum is, in fact, a point of local maximum; so (21) holds. The details follow.

It is convenient to introduce the notation

$$D_1(\xi):=\int_{-1}^{1}(1-x)^{jp}(1+x)^{kp}(1+\alpha(\xi)x)^{p-1}\times x\,dx,$$

$$D_0(\xi):=\int_{-1}^{1}(1-x)^{jp}(1+x)^{kp}(1+\alpha(\xi)x)^{p-1}\times 1\,dx,$$

and

$$D(\xi):=\int_{-1}^{1}(1-x)^{jp}(1+x)^{kp}(1+\alpha(\xi)x)^{p}\,dx.$$

Then

$$\frac{\Phi'_p(\xi)}{\Phi_p(\xi)} = p\alpha'(\xi)\left\{\frac{D_1(\xi)}{D(\xi)} - \frac{\xi}{1+\alpha(\xi)\xi}\right\}.$$

So

$$\frac{D_1(\xi)}{D(\xi)} = \frac{\xi}{1+\alpha(\xi)\xi} \qquad \text{if} \quad \xi \in \{\mathfrak{c}\}_{n,k}. \tag{27}$$

Taking (27) into account it is easily seen that if $\xi \in \{\mathfrak{c}\}_{n,k}$, then

$$\begin{aligned}\frac{\Phi''_p(\xi)}{\Phi_p(\xi)} &= \frac{\Phi''_p(\xi)}{\Phi_p(\xi)} - \left\{\frac{\Phi'_p(\xi)}{\Phi_p(\xi)}\right\}^2\\ &= p\alpha'(\xi)\left\{\frac{D'_1(\xi)}{D(\xi)} - \frac{D_1(\xi)\,D'(\xi)}{(D(\xi))^2} - \frac{1-\alpha'(\xi)\,\xi^2}{(1+\alpha(\xi)\xi)^2}\right\}.\end{aligned} \tag{28}$$

Clearly, $D(\xi) - D_0(\xi) = \alpha(\xi)\,D_1(\xi)$; hence if $\xi \in \{\mathfrak{c}\}_{n,k}$, then

$$\frac{D_0(\xi)}{D(\xi)} = \frac{D(\xi) - \alpha(\xi)\,D_1(\xi)}{D(\xi)} = 1 - \alpha(\xi)\,\frac{\xi}{1+\alpha(\xi)\xi} = \frac{1}{1+\alpha(\xi)\xi}. \tag{29}$$

Now the case $p = 1$ has to be treated separately from the much harder case $p \neq 1$.

LEMMA 7. (21) *holds for* $p = 1$.

Proof. Using Lemma 6 we obtain

$$\begin{aligned}\int_{-1}^{1} (1-x)^j\,(1+x)^k\,x\,dx &= \int_{-1}^{1} (1-x)^j\,\{(1+x)^{k+1} - (1+x)^k\}\,dx\\ &= 2^{j+k+1}\,\frac{\Gamma(j+1)\,\Gamma(k+1)}{\Gamma(j+k+2)}\,\frac{k-j}{j+k+2},\end{aligned}$$

and

$$\begin{aligned}&\int_{-1}^{1} (1-x)^j\,(1+x)^k\,(1+\alpha(\xi)x)\,dx\\ &\quad = 2^{j+k+1}\,\frac{\Gamma(j+1)\,\Gamma(k+1)}{\Gamma(j+k+1)}\left\{1 + \frac{(k-j)\,\alpha(\xi)}{j+k+2}\right\}.\end{aligned}$$

Hence by (22),

$$\frac{\Phi_1'(\xi)}{\Phi_1(\xi)}=\alpha'(\xi)\left\{\frac{k-j}{j+k+2+(k-j)\,\alpha(\xi)}-\frac{\xi}{1+\alpha(\xi)\,\xi}\right\}$$

$$=\alpha'(\xi)\,\frac{(k-j)-(j+k+2)\,\xi}{(j+k+2+(k-j)\,\alpha(\xi))(1+\alpha(\xi)\,\xi)},$$

which shows that Φ_1 has one and only one critical point $\hat{\xi}:=(k-j)/(j+k+2)$ in (ξ_1, ξ_2). In view of (24) and (26) it must be a point of local maximum. Thus, (21) holds.

In order to prove (21) when $p \neq 1$ we need the following representation for $D_1'(\xi)$.

LEMMA 8. *If* $\xi \in (\xi_1, \xi_2)$, $\xi \neq \xi^*$, *then for* $p \in (0, \infty)\backslash\{1\}$ *we have*

$$D_1'(\xi)=(p-1)\,\alpha'(\xi)\{A_1(\xi)+A_2(\xi)+A_3(\xi)\},$$

where

$$A_1(\xi):=\frac{1}{2(1-\alpha(\xi))}\,(D_0(\xi)-D_1(\xi)),$$

$$A_2(\xi):=\frac{1}{2(1+\alpha(\xi))}\,(D_0(\xi)+D_1(\xi)),$$

and

$$A_3(\xi):=\frac{1}{(p-1)\,\alpha(\xi)(1-\alpha(\xi))(1+\alpha(\xi))}\,\{(k-j)\,pD_0(\xi)$$
$$-((j+k)\,p+2)\,D_1(\xi)\}.$$

Proof. Note that $0<|\alpha(\xi)|<1$ since $\xi \in (\xi_1, \xi_2)\backslash\{\xi^*\}$. Using Lagrange interpolation in the points -1, $+1$ and $-1/\alpha=-1/\alpha(\xi)$ where $\xi \neq \xi^*$, we can write

$$x^2=\frac{1}{2(1-\alpha)}\,(1-x)(1+\alpha x)+\frac{1}{2(1+\alpha)}\,(1+x)(1+\alpha x)$$
$$-\frac{1}{(1-\alpha)(1+\alpha)}\,(1-x^2).$$

Clearly, this formula also holds for $\xi = \xi^*$, i.e., when $\alpha(\xi) = 0$. Hence

$$D_1'(\xi) = (p-1)\,\alpha'(\xi) \int_{-1}^{1} (1-x)^{jp}\,(1+x)^{kp}\,(1+\alpha(\xi)\,x)^{p-2}\,x^2\,dx$$
$$= (p-1)\,\alpha'(\xi)\{A_1(\xi) + A_2(\xi) + A_3(\xi)\},$$

where

$$A_1(\xi) := \frac{1}{2(1-\alpha(\xi))} \int_{-1}^{1} (1-x)^{jp}\,(1+x)^{kp}\,(1+\alpha(\xi)x)^{p-1}\,(1-x)\,dx$$
$$= \frac{1}{2(1-\alpha(\xi))}\,(D_0(\xi) - D_1(\xi)),$$
$$A_2(\xi) := \frac{1}{2(1+\alpha(\xi))} \int_{-1}^{1} (1-x)^{jp}\,(1+x)^{kp}\,(1+\alpha(\xi)\,x)^{p-1}\,(1+x)\,dx$$
$$= \frac{1}{2(1+\alpha(\xi))}\,(D_0(\xi) + D_1(\xi)),$$

and

$$A_3(\xi) := -\frac{1}{(p-1)\,\alpha(1-\alpha)(1+\alpha)}$$
$$\times \int_{-1}^{1} (1-x)^{jp+1}\,(1+x)^{kp+1}\,(p-1)\,\alpha(1+\alpha x)^{p-2}\,dx$$
$$= \frac{1}{(p-1)\,\alpha(1-\alpha)(1+\alpha)} \int_{-1}^{1} (1-x)^{jp}\,(1+x)^{kp}\,(1+\alpha x)^{p-1}$$
$$\times \{-(jp+1)(1+x) + (kp+1)(1-x)\}\,dx$$
$$= \frac{1}{(p-1)\,\alpha(1-\alpha)(1+\alpha)} \{(k-j)\,pD_0(\xi) - ((j+k)\,p+2)\,D_1(\xi)\};$$

i.e., Lemma 8 holds.

If $\xi \in \{\mathfrak{c}\}_{n,k}$, then we may use (27) along with (29) to conclude that if $p \in (0, \infty)\backslash\{1\}$, then

$$\frac{A_1(\xi)}{D(\xi)} = \frac{1}{2(1-\alpha(\xi))}\,\frac{D_0(\xi) - D_1(\xi)}{D(\xi)} = \frac{1}{2(1-\alpha(\xi))}\,\frac{1-\xi}{1+\alpha(\xi)\,\xi},$$
$$\frac{A_2(\xi)}{D(\xi)} = \frac{1}{2(1+\alpha(\xi))}\,\frac{D_0(\xi) + D_1(\xi)}{D(\xi)} = \frac{1}{2(1+\alpha(\xi))}\,\frac{1+\xi}{1+\alpha(\xi)\,\xi},$$

$$\frac{A_3(\xi)}{D(\xi)}=\frac{1}{(p-1)\,\alpha(\xi)(1-\alpha(\xi))(1+\alpha(\xi))}\,\frac{(k-j)\,p-((j+k)\,p+2)\,\xi}{1+\alpha(\xi)\,\xi}$$
$$=\frac{1}{(p-1)\,\alpha(\xi)(1-\alpha^2(\xi))}\left\{-\frac{(1-\xi^2)\,\alpha(\xi)\,p}{(1+\alpha(\xi)\,\xi)^2}-\frac{2\xi}{1+\alpha(\xi)\,\xi}\right\}$$
$$=\frac{1}{(p-1)(1-\alpha^2(\xi))(1+\alpha(\xi)\,\xi)^2}\left\{-(1-\xi^2)\,p-2\xi^2-2\,\frac{\xi}{\alpha(\xi)}\right\},$$

since

$$\frac{(1-\xi^2)\,\alpha(\xi)}{1+\xi\alpha(\xi)}=(j+k)\,\xi-(k-j).$$

Hence by Lemma 8,

$$\frac{D_1'(\xi)}{D(\xi)}=(p-1)\,\alpha'(\xi)\left\{\frac{1-\xi}{2(1-\alpha(\xi))(1+\alpha(\xi)\,\xi)}\right.$$
$$\left.+\frac{1+\xi}{2(1+\alpha(\xi))(1+\alpha(\xi)\,\xi)}-\frac{(1-\xi^2)\,p+2\xi^2+2\xi/\alpha(\xi)}{(p-1)(1-\alpha^2(\xi))(1+\alpha(\xi)\,\xi)^2}\right\}. \tag{30}$$

It is clear that $D'(\xi)=p\alpha'(\xi)\,D_1(\xi)$ and so if $\xi\in\{\mathfrak{c}\}_{n,k}$, then by (27),

$$\frac{D_1(\xi)\,D'(\xi)}{(D(\xi))^2}=p\alpha'(\xi)\left(\frac{D_1(\xi)}{D(\xi)}\right)^2=p\alpha'(\xi)\,\frac{\xi^2}{(1+\alpha(\xi)\,\xi)^2}. \tag{31}$$

Using (30) and (31) in (28) we conclude that if $\xi\in\{\mathfrak{c}\}_{n,k}$, then for all $p\in(0,\infty)\backslash\{1\}$ we have

$$\frac{\Phi_p''(\xi)}{\Phi_p(\xi)}=p\alpha'(\xi)\left\{\alpha'(\xi)\left(\frac{(p-1)(1-\xi)}{2(1-\alpha)(1+\alpha\xi)}\right.\right.$$
$$+\frac{(p-1)(1+\xi)}{2(1+\alpha)(1+\alpha\xi)}-\frac{(1-\xi^2)\,p+2\xi^2+2\xi/\alpha}{(1-\alpha)(1+\alpha)(1+\alpha\xi)^2}$$
$$\left.\left.-p\,\frac{\xi^2}{(1+\alpha\xi)^2}+\frac{\xi^2}{(1+\alpha\xi)^2}\right)-\frac{1}{(1+\alpha\xi)^2}\right\}$$
$$=\frac{p\alpha'(\xi)}{(1-\alpha^2)(1+\alpha\xi)^2}\left\{-\alpha'(\xi)(1+\xi^2)-2\xi\,\frac{\alpha'(\xi)}{\alpha(\xi)}-(1-\alpha^2)\right\}.$$

Since

$$\frac{1}{\alpha(\xi)}=\frac{1+(k-j)\xi-(1+j+k)\,\xi^2}{(j+k)\xi-(k-j)}=\frac{1-\xi^2}{(j+k)\,\xi-(k-j)}-\xi,$$

we conclude that

$$\frac{\Phi_p''(\xi)}{\Phi_p(\xi)} = -\frac{p(\alpha'(\xi))^2}{(1-\alpha^2)(1+\alpha\xi)^2} \times \left\{(1-\xi^2)\left(1+\frac{2\xi}{(j+k)\xi-(k-j)}\right)+\frac{1-\alpha^2}{\alpha'(\xi)}\right\} \tag{32}$$

From (12) we deduce that

$$1-\alpha(\xi)=(1+\xi)\,\frac{1+k-j-(1+j+k)\xi}{1+(k-j)\,\xi-(1+j+k)\,\xi^2},$$

$$1+\alpha(\xi)=(1-\xi)\,\frac{1-k+j+(1+j+k)\xi}{1+(k-j)\xi-(1+j+k)\,\xi^2},$$

and

$$\alpha'(\xi)=\frac{(1+j+k)(j+k)\,\xi^2-2(1+j+k)(k-j)\xi+(k-j)^2+j+k}{\{1-(k-j)\xi-(1+j+k)\,\xi^2\}^2}.$$

Hence by Lemma 5,

$$\sigma_{j,k}(\xi) := (1+j+k)(j+k)\,\xi^2-2(1+j+k)(k-j)\xi+(k-j)^2+j+k>0,$$

and for $\xi \in \{\mathfrak{c}\}_{n,k}$ we have

$$\begin{aligned}\frac{\Phi_p''(\xi)}{\Phi_p(\xi)} &= -\frac{p(1-\xi)^2\,(\alpha'(\xi))^2}{(1-\alpha^2)(1+\alpha\xi)^2}\left\{\frac{(2+j+k)\xi-(k-j)}{(j+k)\xi-(k-j)}\right.\\ &\quad\left.+\frac{1-(k-j)^2+2(k-j)(1+j+k)\xi-(1+j+k)^2\,\xi^2}{\sigma_{j,k}(\xi)}\right\}\\ &= -\frac{p(1-\xi^2)(\alpha'(\xi))^2}{\{(1-\alpha)(1+\alpha)(1+\alpha\xi)^2\}\{(j+k)\xi-(k-j)\}}\,\frac{\pi_3(\xi)}{\sigma_{j,k}(\xi)},\end{aligned} \tag{33}$$

where

$$\begin{aligned}\pi_3(\xi) &:= (j+k)(j+k+1)\,\xi^3-3(k-j)(j+k+1)\,\xi^2\\ &\quad+\{3(j+k)(j+k+1)-8jk\}\,\xi-(k-j)(j+k+1).\end{aligned}$$

Note that $\pi_3''(\xi)=6(j+k+1)(j+k)(\xi-\xi^*)$ is negative for $\xi<\xi^*$ and positive for $\xi>\xi^*$; i.e., $\pi_3'(\xi)$ is strictly decreasing on $[\xi_1, \xi^*)$ and strictly increasing on $(\xi^*, \xi_2]$. Since $\pi_3'(\xi^*)=(4jk/(j+k))(j+k+3)>0$ it follows that $\pi_3'(\xi)>0$ for all ξ in $[\xi_1, \xi_2]$. So π_3 can have at most one zero in

$[\xi_1, \xi_2]$. In fact, it does have one zero in (ξ_1, ξ^*). This is seen as follows. The quantity $(j+k)\xi-(k-j)$ is negative for $\xi<\xi^*$ and tends to zero as $\xi \to \xi^*-$. Hence, from (32) and (33) we conclude that $\pi_3(\xi)$ is positive in $(\xi^*-\delta, \xi^*)$ for all small $\delta>0$. The same formulae can be similarly used to conclude that $\pi_3(\xi)$ is negative in $(\xi_1, \xi_1+\delta)$ for all small positive δ. Alternatively, using "Mathematica" (Wolfram Research, Inc.) or by patient calculation we can check that $(j+k)^2\,\pi_3(\xi^*)=8jk(k-j)>0$ for $k>j$, whereas $(j+k+1)^2\,\pi_3(\xi_1)=-8k(j+1)^2<0$. Hence, π_3 must have a zero in (ξ_1, ξ^*) for $k>j$.

Let now $k>j$. If ξ_3 denotes the only zero of π_3 in (ξ_1, ξ^*) then π_3 is negative on $[\xi_1, \xi_3)$ and positive on $(\xi_3, \xi^*]$. From (32) and (33) we see that at any zero of Φ_p' which lies in (ξ_1, ξ^*), the sign of $\Phi_p''(\xi)$ is the same as the sign of $\pi_3(\xi)$. Thus, $\Phi_p''(\xi)$ is negative at each ξ belonging to $\{\mathfrak{c}\}_{n,k} \cap (\xi_1, \xi_3)$ and positive at any ξ that belongs to $\{\mathfrak{c}\}_{n,k} \cap (\xi_3, \xi^*)$. From (24) and (25) it follows that Φ_p has at least one critical point in (ξ_1, ξ^*) if $j<k$. If such a point lies in (ξ_1, ξ_3), then it must be a point of local maximum for Φ_p. Since each point in $\{\mathfrak{c}\}_{n,k} \cap (\xi_1, \xi_3)$ can only be a point of local maximum there can be at most one critical point of Φ_p in (ξ_1, ξ_3). Indeed, two local maxima are separated by a local minimum. If Φ_p has a critical point ξ' which lies in (ξ_3, ξ^*) then *it must be a point of local minimum for* Φ_p. Hence $\Phi_p'(\xi)$ should be positive in $(\xi', \xi'+\delta')$ for some $\delta'>0$. In view of (25), $\Phi_p'(\xi)$ must have at least one zero in (ξ', ξ^*), too, which can only be a point of local minimum, since $\Phi_p''(\xi)>0$ at all the points in $\{\mathfrak{c}\}_{n,k} \cap (\xi_3, \xi^*)$. But, then there must be a point of local maximum between the two local minima, which is a contradiction. So, Φ_p *does not really have a critical point in* (ξ_3, ξ^*). From (32) it follows that $\Phi_p''(\xi)<0$ for all ξ in $\{\mathfrak{c}\}_{n,k} \cap (\xi^*, \xi_2)$. So, *any critical point of* Φ_p *in* (ξ^*, ξ_2) *must be a local maximum.* But if such a point ξ'' existed, $\Phi_p'(\xi)$ would be positive in $(\xi''-\delta'', \xi'')$ for some $\delta''>0$. In view of (25), there would then be a zero of Φ_p' in (ξ^*, ξ'') if $j<k$. This zero of Φ_p' would again be a point of local maximum and we are led to a contradiction. So, Φ_p has no critical point in $[\xi^*, \xi_2]$ if $j<k$. As it has been pointed out earlier, Φ_p' must, because of (24) and (26), vanish at least once in (ξ_1, ξ_2). The above argument shows that it cannot do so in (ξ_3, ξ_2) if $j<k$; but it may vanish in (ξ_1, ξ_3), though not more than once since a zero of Φ_p' in this interval is necessarily a point of local maximum for Φ_p. Summarizing the above discussion we have noted that

(i) Φ_p' has at least one zero in $(\xi_1, \xi_3]$;

(ii) Φ_p' has at most one zero in (ξ_1, ξ_3);

(iii) all the zeros of Φ_p' in (ξ_1, ξ_3) are simple, i.e., $\Phi_p''\neq 0$ if $\Phi_p'=0$;

(iv) Φ'_p has no zero in (ξ_3, ξ_2);

(v) $\Phi'_p(\xi_1+) > 0$, $\Phi'_p(\xi_2-) < 0$ and if $k > j$, then $\Phi'_p(\xi^*) < 0$.

Now let us suppose that Φ'_p has a zero in (ξ_1, ξ_3), call it $\hat{\xi}$. From (22) and the definition of Φ_p given in Remark 4 it can be concluded with the help of a known result [4, Sect. 5.51] that Φ'_p is an analytic function of (the complex variable ξ) in a small neighbourhood of the point ξ_3. This implies that Φ'_p can only have a zero of finite multiplicity at ξ_3. From (v) and (iii) it follows that $\Phi'_p(\xi) > 0$ for $\xi_1 < \xi < \hat{\xi}$ and $\Phi'_p(\xi) < 0$ for $\hat{\xi} < \xi < \xi_3$. Since $\Phi'_p(\xi^*) < 0$, (iv) implies that ξ_3, if it is a zero of Φ'_p, must be of even multiplicity, so that $\Phi'_p(\xi) < 0$ for $\xi_3 < \xi \leqslant \xi_2$. The conclusion is that, in this case, the function $\Phi_p(\xi)$ is strictly increasing on $(\xi_1, \hat{\xi})$ and strictly decreasing on $(\hat{\xi}, \xi_2)$; i.e., (21) holds.

The other possibility is that Φ'_p has no zero in (ξ_1, ξ_3).Then it must have a zero at ξ_3. Since $\Phi'_p(\xi^*) < 0$ it follows from (iv) that the zero of Φ'_p at ξ_3 must be of odd multiplicity. So, in this case $\Phi'_p(\xi)$ is strictly increasing on (ξ_1, ξ_3) and strictly decreasing on (ξ_3, ξ_2); i.e., (21) holds again.

If $j = k$, then $\xi_1 = -\xi_2$ and $\xi^* = 0$. According to (25), $\Phi'_p(0) = 0$. Furthermore, in this case, formulae (32) and (33) reduce to

$$\frac{\Phi''_p(\xi)}{\Phi_p(\xi)} = -\frac{p(\alpha'(\xi))^2}{(1-\alpha^2)(1+\alpha\xi)^2}\left\{(1-\xi^2)\left(\frac{k+1}{k}\right) + \frac{1-\alpha^2}{\alpha'(\xi)}\right\}$$

and

$$\frac{\Phi''_p(\xi)}{\Phi_p(\xi)} = -\frac{p(1-\xi^2)(\alpha'(\xi))^2\,\{k(2k+1)\,\xi^2 + k(2k+3)\}}{2k\{k(2k+1)\,\xi^2 + k\}\,\{(1-\alpha)(1+\alpha)(1+\alpha\xi)^2\}},$$

respectively. Hence, $\Phi''_p(\xi) < 0$ if ξ is a critical point of Φ_p lying in (ξ_1, ξ_2). Taking also into account that Φ_p is even, no point of $(\xi_1, 0)$ or of $(0, \xi_2)$ can be a zero of Φ'_p. Since Φ_p must have a critical point in (ξ_1, ξ_2) it (the critical point) must lie at $\xi = 0 = \xi^*$ and it must be a point of local as well as global maximum for Φ_p.

Next we show that for all $p \in (0, \infty)$,

$$\Phi_p(\xi_{1,n,k}) > \Phi_p(\xi_{2,n,k}) \qquad \text{if} \quad k > j = n-1-k. \tag{34}$$

To start with we observe that

$$\frac{\Phi_p(\xi_{1,n,k})}{\Phi_p(\xi_{2,n,k})} = \frac{j^{jp}(k+1)^{(k+1)\,p}}{(j+1)^{(j+1)\,p}k^{kp}}\,\frac{\Gamma((j+1)\,p+1)\,\Gamma(kp+1)}{\Gamma(jp+1)\,\Gamma((k+1)\,p+1)} = \frac{\varphi_p(j)}{\varphi_p(k)},$$

where

$$\varphi_p(x) := \frac{x^{xp}}{(x+1)^{(x+1)\,p}} \frac{\Gamma((x+1)\,p+1)}{\Gamma(xp+1)}.$$

So, (34) follows from the following lemma.

LEMMA 9. *For all $p>0$, the function φ_p is a strictly decreasing function of x on $[1, \infty)$.*

Proof. Clearly,

$$\frac{1}{p}\frac{\varphi_p'(x)}{\varphi_p(x)} = \frac{\Gamma'((x+1)\,p+1)}{\Gamma((x+1)\,p+1)} - \frac{\Gamma'(xp+1)}{\Gamma(xp+1)} - \log\left(1+\frac{1}{x}\right).$$

According to a known formula [4, p. 228, Example 10],

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\frac{1}{z} - \gamma + \sum_{v=1}^{\infty}\left(\frac{1}{v} - \frac{1}{z+v}\right),$$

where γ is the Euler's constant. Hence

$$\frac{1}{p}\frac{\varphi_p'(x)}{\varphi_p(x)} = \frac{1}{xp+1} - \frac{1}{(x+1)\,p+1} + \sum_{v=1}^{\infty}\left\{\frac{1}{v} - \frac{1}{(x+1)\,p+v+1}\right\}$$

$$-\sum_{v=1}^{\infty}\left\{\frac{1}{v} - \frac{1}{xp+v+1}\right\} - \log\left(1+\frac{1}{x}\right)$$

$$= \sum_{v=1}^{\infty}\left\{\frac{1}{xp+v} - \frac{1}{(x+1)\,p+v}\right\} - \log\left(1+\frac{1}{x}\right),$$

since $1/v - 1/((x+1)\,p+v+1) = O(v^{-2})$ as $v \to \infty$.

Now we note that $1/(xp+t) - 1/((x+1)\,p+t)$ is a positive decreasing function of t and hence for all $v \in \mathbb{N}$,

$$\frac{1}{xp+v} - \frac{1}{(x+1)\,p+v} < \int_{v-1}^{v}\left\{\frac{1}{xp+t} - \frac{1}{(x+1)\,p+t}\right\}dt.$$

Thus

$$\frac{1}{p}\frac{\varphi_p'(x)}{\varphi_p(x)} < \int_0^\infty \left\{\frac{1}{xp+t} - \frac{1}{(x+1)\,p+t}\right\} dt - \log\left(1+\frac{1}{x}\right)$$

$$= \lim_{T\to\infty} \int_0^T \left\{\frac{1}{xp+t} - \frac{1}{(x+1)\,p+t}\right\} dt - \log\left(1+\frac{1}{x}\right)$$

$$= \lim_{T\to\infty} \log\left(\frac{xp+T}{(x+1)\,p+T}\right) = 0.$$

Lemma 9 is proved and so is (34).

The final step. We have shown that if $k \geqslant (n-1)/2$ and $0 < p < \infty$, then

$$\min_{\xi_{1,\,n,\,k} \leqslant \xi \leqslant \xi_{2,\,n,\,k}} \Phi_p(\xi) = \Phi_p(\xi_{2,\,n,\,k}).$$

Since $\xi_{2,\,n,\,k} = \xi_{1,\,n,\,k+1}$ it follows that if $k > j = n-1-k$, then

$$\min_{\xi \in I_{n,\,k+1}} \Phi_p(\xi) < \min_{\xi \in I_{n,\,k}} \Phi_p(\xi)$$

and so for $1 \leqslant \mu \leqslant [n/2]$ and $p > 0$,

$$\min_{-1+(2\mu/n) \leqslant \xi \leqslant 1-(2\mu/n)} \Phi_p(\xi) = \Phi_p\left(1 - \frac{2\mu}{n}\right)$$

$$= \left(\frac{n^n}{\mu^\mu (n-\mu)^{n-\mu}}\right)^p 2\,\frac{\Gamma(\mu p+1)\,\Gamma((n-\mu)\,p+1)}{\Gamma(pn+2)}.$$

In particular,

$$\min_{-1+2/n \leqslant \xi \leqslant 1-2/n} \Phi_p(\xi) = \Phi_p\left(1-\frac{2}{n}\right)$$

$$= \left(\frac{n^n}{2^n (n-1)^{n-1}}\right)^p \int_{-1}^1 (1-x)^p\,(1+x)^{(n-1)\,p}\,dx$$

$$= \left(\frac{n^n}{(n-1)^{n-1}}\right)^p 2\,\frac{\Gamma(p+1)\,\Gamma((n-1)\,p+1)}{\Gamma(pn+2)}.$$

As indicated earlier, $\mathfrak{M}_{n,\,\mu,\,p,\,\xi} = (2^{-1}\,\Phi_p(\xi))^{1/p}$ and so recalling that

$$\mathfrak{M}_{n,\,\mu,\,p,} = \inf_{-1+2\mu/n \leqslant \xi \leqslant 1-2\mu/n} \mathfrak{M}_{n,\,\mu,\,p,\,\xi}$$

we obtain Theorem 1 for all $p > 0$ and $\mu \geqslant 1$.

2.3. *The Case $p=0$ and $\mu \geqslant 1$ of Theorem* 1

Now let $p=0$. From the case $0<p<\infty$, which has already been settled, it follows that if $f \in \mathcal{P}_{n,\mu}$, then

$$\|f\|_0 := \exp\left(\frac{1}{2}\int_{-1}^{1} \log|f(x)|\,dx\right) \geqslant \frac{n^n}{e^n \mu^\mu (n-\mu)^{n-\mu}}\,\|f\|_\infty,$$

wherein equality holds for all polynomials of the form $c(1+x)^{n-\mu}(1-x)^\mu$ and $c(1+x)^\mu(1-x)^{n-\mu}$. However, having proved it by a limiting process we cannot claim that the inequality is strict for all other polynomials belonging to $\mathcal{P}_{n,\mu}$. But this is true and can be seen as follows.

For $\xi \in I_{n,k}$, let

$$\omega_{0,k}(\xi) := \int_{-1}^{1} \log\left|\frac{(1-x)^j(1+x)^k(1+\alpha(\xi)x)}{(1-\xi)^j(1+\xi)^k(1+\alpha(\xi)\xi)}\right|dx,$$

where $j=n-1-k$ and $\alpha(\xi)$ is as in (12). The information given in Remark 5 shows that

$$\mathfrak{M}_{n,\mu,0,\xi} = \exp(\tfrac{1}{2}\omega_{0,k}(\xi)), \qquad (\xi \in I_{n,k}).$$

Using the formula for $\alpha(\xi)$ given in (12) we see that for $\xi \in (\xi_{1,n,k}, \xi_{2,n,k})$ we have

$$\omega'_{0,k}(\xi) = \alpha'(\xi)\left\{\int_{-1}^{1} \frac{x}{1+\alpha(\xi)x}\,dx - \frac{2\xi}{1+\alpha(\xi)\xi}\right\}.$$

Simple calculations show that

$$\omega'_{0,k}(\xi) \to +\infty \qquad \text{as} \quad \xi \to \xi_1+,$$

whereas

$$\omega'_{0,k}(\xi) \to -\infty \qquad \text{as} \quad \xi \to \xi_2-.$$

Furthermore, if $\xi^* = \xi^*_{n,k}$ is as in (20), then

$$\omega'_{0,k}(\xi^*) < 0 \quad \text{if } j<k, \qquad \omega'_{0,k}(\xi^*) = 0 \quad \text{if } j=k.$$

We leave it to the reader to verify that if ξ is a critical point of $\omega_{0,k}$ in (ξ_1, ξ_2), i.e., if

$$\int_{-1}^{1} \frac{x}{1+\alpha(\xi)x}\,dx = \frac{2\xi}{1+\alpha(\xi)\xi},$$

then

$$\begin{aligned}\omega''_{0,k}(\xi) &= \frac{2\alpha'(\xi)}{(1-\alpha^2)(1+\alpha(\xi)\xi)^2}\\ &\quad\times\left\{-\alpha'(\xi)(1+\xi^2)-2\xi\,\frac{\alpha'(\xi)}{\alpha(\xi)}-(1-\alpha^2(\xi))\right\}\\ &= -\frac{2(\alpha'(\xi))^2}{(1-\alpha^2)(1+\alpha(\xi)\xi)^2}\\ &\quad\times\left\{(1-\xi^2)\left(1+\frac{2\xi}{(j+k)\xi-(k-j)}\right)+\frac{(1-\alpha)(1+\alpha)}{\alpha'(\xi)}\right\}.\end{aligned}\tag{35}$$

Compare this with (32). Imitating that part of the proof of Theorem 1 (in the case $p>0$), which follows formula (32), we arrive at the conclusion that for $1\leqslant\mu\leqslant[n/2]$

$$\mathfrak{M}_{n,\mu,0}=\mathfrak{M}_{n,\mu,0,\xi}$$

if and only if $\xi=\pm(1-(2\mu/n))$. Now, some fairly simple calculations lead us to the proof of Theorem 1 in the remaining case $p=0$.

2.4. *The Case $\mu=0$ of Theorem* 1

Now we consider the case $\mu=0$. From Remark 5 it follows that

$$\mathfrak{M}_{n,0,p}=\min\{\inf_{0\leqslant\xi\leqslant 1-2/n}\mathfrak{M}_{n,1,p,\xi},\ \inf_{1-2/n\leqslant\xi<1}\mathfrak{M}_{n,0,p,\xi},\ \mathfrak{M}_{n,0,p,1}\}.$$

Let us determine $\min\{\inf_{1-2/n\leqslant\xi<1}\mathfrak{M}_{n,0,p,\xi},\mathfrak{M}_{n,0,p,1}\}$. In view of Remark 5, we have

$$\inf_{1-2/n\leqslant\xi<1}\mathfrak{M}_{n,0,p,\xi}=\inf_{1-2/n\leqslant\xi<1}\|P_{n,n-1,\xi}\|_p=\min_{1-2/n\leqslant\xi\leqslant 1}\|P_{n,n-1,\xi}\|_p.$$

First let $0<p<\infty$ and extend the definition of $\Phi_p(\xi)$ to values of ξ in $(1-2/n,1]$. Note that $k=n-1$. Thus, for all ξ in $[1-2/n,1]$ and all $p>0$,

$$\begin{aligned}\Phi_p(\xi) &:= 2\,\|P_{n,n-1,\xi}\|_p^p\\ &= 2\int_{-1}^{1}\left|\frac{(1+x)^{n-1}\,(1+\alpha(\xi)x)}{(1+\xi)^{n-1}\,(1+\alpha(\xi)\xi)}\right|^p dx \qquad \left(\alpha(\xi):=-\frac{n-1}{n\xi+1}\right).\end{aligned}$$

The formula (22) for $\Phi_p'(\xi)/\Phi_p(\xi)$ remains valid. It shows that

$$\Phi_p'\left(1-\frac{2}{n}+\right)=\frac{n}{2(n-1)}>0$$

and so $\Phi_p'(\xi)$ increases with ξ in the immediate neighbourhood of $1-2/n$.

As in the proof of the case $\mu \geqslant 1$, we see that Φ_1 has one and only one critical point in $(1-2/n, 1)$, which lies at $(n-1)/(n+1)$. So,

$$\inf_{1-2/n \leqslant \xi \leqslant 1} \Phi_1(\xi)=\min\left\{\Phi_1\left(1-\frac{2}{n}\right), \Phi_1(1)\right\}.$$

Let $p \in (0, \infty)\backslash 1$ and let $\{\mathfrak{c}\}_{n, n-1}$ denote the critical points of Φ_p in $(1-2/n, 1)$. Formula (32) which gives the value of $\Phi_p''(\xi)/\Phi_p(\xi)$ at each point $\xi \in \{\mathfrak{c}\}_{n, n-1}$ remains valid and gives

$$\frac{\Phi_p''(\xi)}{\Phi_p(\xi)}=-\frac{p(\alpha'(\xi))^2}{(1-\alpha^2(\xi))(1+\alpha(\xi)\,\xi)^2}\,\frac{1-\xi^2}{n-1};$$

i.e., $\Phi_p''(\xi)$ is negative at all the critical points of Φ_p which lie in $(1-2/n, 1)$. This means that any local extremum of Φ_p in $(1-2/n, 1)$ can only be a local maximum. Hence,

$$\inf_{1-2/n \leqslant \xi \leqslant 1} \Phi_p(\xi)=\min\left\{\Phi_p\left(1-\frac{2}{n}\right), \Phi_p(1)\right\}$$

for all $p \in (0, \infty)$; i.e.,

$$\inf_{1-2/n \leqslant \xi < 1} \mathfrak{M}_{n, 0, p, \xi}=\min\left\{\left(\frac{1}{2}\,\Phi_p\left(1-\frac{2}{n}\right)\right)^{1/p}, \left(\frac{1}{2}\,\Phi_p(1)\right)^{1/p}\right\}.$$

As shown earlier (see the discussion following Remark 4), $(2^{-1}\Phi_p(1))^{1/p} > \|P^*_{n, n-1, 1}\|_p$; so

$$\begin{aligned}\min\{\inf_{1-2/n \leqslant \xi < 1} \mathfrak{M}_{n, 0, p, \xi}, \mathfrak{M}_{n, 0, p, 1}\}^p &= \min\left\{\frac{1}{2}\,\Phi_p\left(1-\frac{2}{n}\right), \|P^*_{n, n-1, 1}\|_p^p\right\}\\ &= \min\{\|q_{n, n-1, *}\|_p^p, \|P^*_{n, n-1, 1}\|_p^p\}.\end{aligned}$$

It follows from Lemma 6 that

$$\frac{\|q_{n, n-1, *}\|_p^p}{\|P^*_{n, n-1, 1}\|_p^p}=\frac{n^{np}}{(n-1)^{(n-1)\,p}}\,\frac{\Gamma((n-1)\,p+1)\,\Gamma(p+1)}{\Gamma(np+1)},$$

which, we claim, is larger than 1. This is because

$$\frac{\Gamma(xp+1)}{\Gamma((x-1)\,p+1)}\,\frac{(x-1)^{(x-1)\,p}}{x^{xp}}<\Gamma(p+1) \tag{36}$$

for all $x>1$. Indeed, if $\Delta(x)$ denotes the left-hand side of (36), then, using the formula for $\Gamma'(z)/\Gamma(z)$ mentioned earlier, we get

$$\begin{aligned}\frac{1}{p}\,\frac{\Delta'(x)}{\Delta(x)}&=\frac{\Gamma'(xp+1)}{\Gamma(xp+1)}-\frac{\Gamma'((x-1)\,p+1)}{\Gamma((x-1)\,p+1)}+\log\frac{x-1}{x}\\&=\sum_{\nu=1}^{\infty}\left\{\frac{1}{(x-1)\,p+\nu}-\frac{1}{xp+\nu}\right\}+\log\frac{x-1}{x}\\&<\int_0^{\infty}\left\{\frac{1}{(x-1)\,p+t}-\frac{1}{xp+t}\right\}dt+\log\frac{x-1}{x},\end{aligned}$$

since $\{1/((x-1)\,p+t)-1/(xp+t)\}$ is a positive decreasing function of t. Thus,

$$\frac{1}{p}\,\frac{\Delta'(x)}{\Delta(x)}<\lim_{T\to\infty}\int_0^{T}\left\{\frac{1}{(x-1)\,p+t}-\frac{1}{xp+t}\right\}dt+\log\frac{x-1}{x}=0,$$

which proves (24). Hence,

$$\min\{\inf_{1-2/n\leqslant\xi<1}\mathfrak{M}_{n,\,0,\,p,\,\xi},\,\mathfrak{M}_{n,\,0,\,p,\,1}\}=\left(\frac{1}{2}\int_{-1}^{1}\left(\frac{1+x}{2}\right)^{np}dx\right)^{1/p}.$$

In the course of the above argument we have also shown that

$$\inf_{0\leqslant\xi\leqslant 1-2/n}\mathfrak{M}_{n,\,1,\,p,\,\xi}>\min\{\inf_{1-2/n\leqslant\xi<1}\mathfrak{M}_{n,\,0,\,p,\,\xi},\,\mathfrak{M}_{n,\,0,\,p,\,1}\};$$

so

$$\mathfrak{M}_{n,\,0,\,p}=\left(\frac{1}{2}\int_{-1}^{1}\left(\frac{1+x}{2}\right)^{np}dx\right)^{1/p}\qquad(0<p<\infty).$$

Equivalently, for each $f\in\mathscr{P}_{n,\,0}\equiv\mathscr{P}_n$,

$$\|f\|_\infty\leqslant(np+1)^{1/p}\,\|f\|_p, \tag{37}$$

where we have an equality only for constant multiples of $q_{n,\,0}$ or of $q_{n,\,n}$. This proves Theorem 1 in the case $\mu=0$ and $p>0$.

Letting p tend to zero in (37) we conclude that for all $f \in \mathscr{P}_{n,0}$, we have

$$\|f\|_\infty \leqslant e^n \|f\|_0,$$

wherein equality holds for polynomials of the form $c(1+x)^n$ and $c(1-x)^n$. For other polynomials in $\mathscr{P}_{n,0} \equiv \mathscr{P}_n$, the inequality is strict; that can be proved the way we identified the extremal polynomials in the case $p=0$ and $\mu \geqslant 1$. Little new is involved; we leave the details to the reader.

3. PROOF OF COROLLARY 1

Let f be a polynomial of degree at most n having no zero in the open unit disk. Suppose in addition, that f has zeros of multiplicity at least μ at -1 and 1 where $0 \leqslant \mu \leqslant [n/2]$. Then $F(z) := f(z)\,\overline{f(\bar{z})}$ is a polynomial of degree at most $2n$ with real coefficients and having no zeros in the open unit disk. Besides, F has zeros of multiplicity at least 2μ at -1 and 1. Hence, by Theorem 1,

$$\|F\|_{p/2} > \|q_{2n,\,2\mu,\,*}\|_{p/2}\,\|F\|_\infty, \qquad 0 \leqslant p < \infty, \tag{39}$$

unless F is a constant multiple of $q_{2n,\,2\mu}$ or $q_{2n,\,2n-2\mu}$. However, F can be a constant multiple of $q_{2n,\,2\mu}$ or $q_{2n,\,2n-2\mu}$ only if f is a constant (possibly non-real) multiple of $q_{n,\,\mu}$ or of $q_{n,\,n-\mu}$. From this, Corollary 1 follows since

$$\|F\|_{p/2} = \|f\|_p^2, \qquad \|q_{2n,\,2\mu,\,0}\|_{p/2} = \|q_{n,\,\mu,\,0}\|_p^2, \qquad \text{and} \qquad \|F\|_\infty = \|f\|_\infty^2.$$

4. PROOF OF COROLLARY 2

According to Theorem 1, if f or $-f$ belongs to $\mathscr{P}_{n,1}$, then

$$\|f\|_\infty \leqslant \frac{(n-1)^{n-1}}{n^n} \left(\frac{\Gamma(pn+2)}{\Gamma(pn-p+1)\,\Gamma(p+1)} \right)^{1/p} \|f\|_p,$$

where equality holds only for constant multiples of $q_{n,\,1}$ or of $q_{n,\,n-1}$.

Corollary 2 follows by combining this result with another result according to which if f or $-f$ belongs to $\mathscr{P}_{n,0}$, then [3, p. 205, Corollary 1] (also see [9])

$$\|f'\|_\infty \leqslant \frac{1}{2} \frac{n^n}{(n-1)^{n-1}} \|f\|_\infty,$$

with equality only for constant multiples of $q_{n,\,1}$ or of $q_{n,\,n-1}$.

5. FINAL REMARKS

It is not without interest that our inequalities are valid and also sharp for all $p \geqslant 0$. The case $p \in [0, 1)$ usually presents difficulties because $\|\cdot\|_p$ ceases to be a norm for such values of p. This point is well illustrated by the paper [2].

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